

# ROUND SURGERY AND CONTACT STRUCTURES ON 3-MANIFOLDS

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**ABSTRACT.** Contact round surgery of contact 3-manifolds is introduced in this paper. By using this method, an alternative proof of the existence of a contact structure on any closed orientable 3-manifold is given. It is also proved that any contact structure on any closed orientable 3-manifold is constructed from the standard contact structure on the 3-dimensional sphere by contact round surgeries. For the proof, important operations in contact topology, the Lutz twist and the Giroux torsion, are described in terms of contact round surgeries.

## 1. INTRODUCTION

Surgery is a basic and important method in studying topology of manifolds. Not only for manifolds themselves but also for geometric structures on manifolds, surgery is important. In this paper, a new notion of round surgery with contact structures is introduced. It is proved that all closed connected contact 3-manifolds are constructed by this method. It would give a new perspective on the study of contact structures.

Round surgery of a manifold was originally introduced by Asimov [As]. He introduced the notion of round handle in order to study non-singular Morse Smale flows (see [As], [Mo]). Further, the theory had been applied to some geometric studies (see [Mi], [EtGh], [V], [Ba]). A *round handle* is roughly defined as a product of an ordinary handle with a circle. In other words, a round handle of index  $k$  of dimension  $n$  is  $R_k = D^k \times D^{n-k-1} \times S^1$  attached to the boundary  $\partial N$  of an  $n$ -dimensional manifold  $N$  by an embedding  $\varphi: \partial D^k \times D^{n-k-1} \times S^1 \rightarrow \partial N$ . Round surgeries of an  $m$ -dimensional manifold  $M$  is defined through cobordisms by attaching round handles of dimension  $m+1$  to  $M \times [0, 1]$  (see Section 3 for precise definition). In this paper, we are mainly dealing with connected 3-dimensional manifolds. Therefore, there exist two kinds of round surgeries, index 1 and index 2. Asimov proved in [As] that any closed orientable 3-manifold is obtained from a 3-dimensional sphere by a sequence of round surgeries of index 1 and index 2.

One of the purposes of this paper is to apply this surgery theory to contact 3-manifolds. A *contact structure* on a 3-dimensional manifold is a completely non-integrable plane field (see Subsection 2.1 for precise definition). It is known that on any closed orientable 3-manifold, there exists a contact structure. It is proved firstly by Martinet ([Mar], [L]), and there are some alternative proofs (see [Ge]).

**Theorem A.** *On any closed orientable 3-manifold, there exists a contact structure.*

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One of the results in this paper is an alternative proof of this existence theorem. By defining contact round surgery, that is, round surgery with contact structure, we apply Asimov's result above. By contact round surgeries, a contact structure is constructed on any closed orientable 3-manifold (see Subsection 6.1).

Not only the existence of a contact structure, we show that all contact structures on all closed orientable 3-manifolds are constructed by contact round surgeries from the standard contact 3-dimensional sphere.

**Theorem B.** *Any contact structure on any closed orientable 3-manifold is constructed from the standard contact 3-dimensional sphere by a sequence of contact round surgeries of index 1 and index 2.*

In the proof of this theorem, all closed connected contact 3-manifolds are reduced to contact structures on  $S^3$  by contact round surgeries (Subsection 6.2). Then the classification of contact structures on  $S^3$  due to Eliashberg ([El1], [El3]) is applied. It is known that contact structures on  $S^3$  are described by the Lutz twists, a modification of contact structures introduced by Lutz [L]. In order to apply the classification result, it is proved that the Lutz twist is represented by a certain sequence of contact round surgeries (see Theorem 5.1).

It should be remarked that the realization of the generalized version of the Lutz twist along a certain torus is given in this paper. It is a distinctive point of this method compared with the Weinstein surgery (see [Wei], [DGe]). Because of this, as well as the Lutz twist, it is proved that the Giroux torsion is closely related to contact round surgeries (see Subsection 5.1). The Giroux torsion is a notion introduced by Giroux [Gi2], which is a source of symplectically non-fillable but tight contact structures.

Some kinds of contact surgeries had appeared in the study of contact structure. We should first mention contact Dehn surgery along transverse links due to Lutz [L] and Martinet [Mar]. By using this surgery method, they proved the existence of a contact structure on any closed orientable 3-manifold (Theorem A).

Symplectic or Stein handlebody surgery was studied by Weinstein [Wei] and Eliashberg [El2]. It is an important tool to study the fillability of contact manifolds by symplectic or Stein manifolds.

Contact Dehn surgery along Legendrian link is now actively studied (see for example [OzbSt]). In terms of this surgery, the symplectic handlebody surgery is written as  $(-1)$ -surgery. Note that the surgery coefficients in this case are measured with respect to contact framings of Legendrian knots. Ding and Geiges [DGe] proved that any rational contact Dehn surgery along a Legendrian knot is represented by contact Dehn surgeries of surgery coefficients  $\pm 1$ . Further, they proved that any contact structure on any closed orientable 3-manifold is constructed by contact  $(\pm 1)$ -surgeries. Etnyre [Et] illuminated the study of  $(+1)$ -surgeries. After Stipsicz [St] related these surgeries to open book decomposition, these surgeries are studied with the Heegaard-Floer theory and the Ozsváth-Szabó contact invariants (see [OzsSz]).

This article is organized as follows. In the next section, we review some basic notions on contact topology which are needed in the discussions later. First, we review some basic definitions and properties, and then some useful results in the convex surface theory. In Section 3, we review the round handle theory. Contact round surgeries are defined in Section 4. In Section 5, a relationship between contact round surgeries and the Lutz twist is discussed. We also mention a relationship with the Giroux torsion. Finally, Theorem A and Theorem B are proved in Section 6.

Several years has passed since the former version was written. Meanwhile, the results in this paper has been considerably improved. Another construction of the Lutz twist is given in [Ad2]. Furthermore, symplectic round handles with the Liouville vector fields corresponding to contact round surgeries are also constructed by the author in any even dimension [Ad1]. Researches on contact structures on higher-dimensional manifolds have advanced (see [MasNWen], [BoElMu]). Applications of contact round surgery to higher-dimensions are expected.

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## 2. CONTACT GEOMETRY

In this section, we introduce basic notions and techniques in contact geometry which are needed in the discussion later. First, we introduce some basic notions. And then, we review convex surface theory which is a useful technique in contact topology.

**2.1. Contact structures on 3-manifolds.** Let us begin with basic definitions. A *contact structure* on a 3-dimensional manifold  $M$  is defined to be a tangent plane field  $\xi$  on  $M$  which is completely non-integrable. In other words, contact structure  $\xi$  is represented locally as the kernel  $\xi = \ker \alpha$  of a 1-form  $\alpha$  which satisfies  $\alpha \wedge d\alpha \neq 0$ . Note that the sign of the 3-form  $\alpha \wedge d\alpha$  does not depend on the choice of the 1-form  $\alpha$  but on  $\xi$  itself. Therefore, a 3-manifold with a contact structure is orientable. In this paper, we assume that 3-manifolds are oriented, and that contact structures are *positive*, that is, each corresponding 3-form is a positive volume form on each oriented manifolds.

Some of the most basic properties of contact structures are the local triviality and the global stability. Any contact structure  $\xi$  on a  $(2n + 1)$ -dimensional manifold is locally equivalent to the standard contact structure  $\xi_0 := \{dz + \sum_{i=1}^n (-y_i dx_i + x_i dy_i) = 0\}$  on  $\mathbb{R}^{2n+1}$ , that is, there exists a local diffeomorphism which maps  $\xi$  to  $\xi_0$  (the Darboux theorem). This implies that there is no local invariant for contact structures. On the other hand, a deformation of a contact structure on a closed manifold can be traced by a one-parameter family of global diffeomorphisms (the Gray theorem). This implies that contact structures are flexible. Here, we should mention the local triviality of contact structures along curves transverse to the structures due to Lutz and Martinet. Let  $\xi_0$  be the standard contact structure on  $S^1 \times \mathbb{R}^{2n}$  defined by the 1-form  $d\varphi + \sum_{i=1}^n (-y_i dx_i + x_i dy_i)$ , where  $\varphi$  is a coordinate of  $S^1$  and  $(x_1, y_1, \dots, x_n, y_n)$  are those of  $\mathbb{R}^{2n}$ .

**Theorem 2.1** (Lutz [L], Martinet [Mar]). *Let  $\gamma$  be an embedded circle in a  $(2n + 1)$ -dimensional manifold  $M$ , and  $\xi$  a contact structure defined on a tubular neighborhood of  $\gamma$ . Assume that  $\xi$  is transverse to  $\gamma$  at any point of  $\gamma$ . Then there exists a local diffeomorphism from a tubular neighborhood of  $\gamma$  to a tubular neighborhood of  $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^2$  which maps  $\gamma$  to  $S^1 \times \{0\}$  and maps  $\xi$  to  $\xi_0$ .*

In contact topology of 3-dimensional manifolds, it is important to know contact structures on neighborhoods of embedded surfaces. We will discuss it later in the next subsection.

On curves transverse to contact structures, the following property is well-known (see for example [Ge]).

**Proposition 2.2.** *For any curve  $L$  in a contact 3-manifold, there exist positively and negatively transverse curves  $L_{\pm}$  which are  $C^0$ -close to  $L$ .*

It is well-known that contact structures on 3-dimensional manifolds are divided into two contradictory classes, tight and overtwisted. A contact structure  $\xi$  on a 3-dimensional manifold  $M$  is said to be *overtwisted* if there exists an embedded disk  $D \subset M$  which is tangent to  $\xi$  along its boundary:  $T_x D = \xi_x$  at any point  $x \in \partial D$ . A contact structure  $\xi$  is said to be *tight* if it is not overtwisted.

Classification of contact structures is an important problem. The classification, up to isotopy, of overtwisted contact structures is obtained by Eliashberg [El1]. It is proved that overtwisted contact structures which are homotopic as plane fields on a 3-dimensional manifold are isotopic. In other words, the classification is reduced to some algebraic arguments in this case. Concerning tight contact structures, although classifications on some manifolds had been obtained, it is still open on many manifolds. In this paper, we need the classification of contact structures on the 3-dimensional sphere  $S^3$ . In this case, it is proved by Eliashberg [El3] that there exists a unique tight contact structure called the standard contact structure on  $S^3$ . Combining these results, we have the complete classification of contact structures on  $S^3$ . With some trivialization, the homotopy classes of tangent plane fields are identified with the class of  $\pi_3(S^2) \cong \mathbb{Z}$ , by taking the Gauss mapping. The classification is as follows.

**Theorem 2.3** (Eliashberg, [El3]). *One homotopy class in  $\pi_3(S^2) \cong \mathbb{Z}$  contains two non-equivalent contact structures on  $S^3$ , tight and overtwisted. Any other class in  $\pi_3(S^2) \cong \mathbb{Z}$  contains a unique overtwisted contact structure on  $S^3$ .*

**2.2. Convex surface theory.** Convex surface theory is one of key tools which caused a breakthrough on 3-dimensional contact topology. We review some basic properties of convex surfaces in contact 3-manifolds which are needed in the definition of contact round surgery and the discussion below.

Before defining convexity of a surface in a contact 3-manifold, we observe 1-dimensional singular foliation on the surface traced by the contact plane field. Let  $F$  be an embedded surface in a contact 3-manifold  $(M, \xi)$ . The contact structure  $\xi$  traces a singular 1-dimensional foliation on  $F$ . In other words,  $\xi_x \cap T_x F$  is a line in  $T_x M$  if  $\xi_x$  and  $T_x F$  do not coincide at  $x \in M$ . Then we obtain a line field  $\mathcal{F}$  with singularities where  $\xi_x = T_x F$ . By integrating the line field, we obtain a 1-dimensional foliation with singularities on  $F$ . Such a foliation is called the *characteristic foliation* on  $F$  with respect to  $\xi$ . Let  $F_{\xi}$  denote it. It is known that a characteristic foliation on a surface determines a germ of contact structures along the surface.

**Proposition 2.4** (Giroux, [Gi1]). *Let  $F$  be an embedded surface in each contact 3-manifold  $(M_i, \xi_i)$ ,  $i = 1, 2$ . Assume that they have the same characteristic foliation:  $F_{\xi_1} = F_{\xi_2}$ . Then there exists a local diffeomorphism between neighborhoods of  $F \subset M_1$  and  $F \subset M_2$  mapping the contact structure  $\xi_1$  to  $\xi_2$  and the surface  $F$  to  $F$ .*

This implies that two contact manifolds with boundaries can be glued together if they have the same characteristic foliation with orientation on their boundaries.

Now we define convexity. It is defined by using the following vector field. A vector field  $X$  on a contact 3-manifold  $(M, \xi)$  is said to be *contact* if its flow  $\varphi_t$  preserves the

contact structure  $\xi$ :  $(\varphi_t)_*\xi = \xi$ . Let  $F$  be a surface embedded in a contact 3-manifold  $(M, \xi)$ . The surface  $F$  is said to be *convex* if there exists a contact vector field on a neighborhood of  $F \subset M$  which is transverse to  $F$ . It is known that any surface in a contact 3-manifold can be approximated to a convex surface.

**Theorem 2.5** (Giroux, [Gil]). *For any embedded surface  $F$  in a contact 3-manifold  $(M, \xi)$ , there exists a convex surface  $\tilde{F} \subset (M, \xi)$  which is  $C^\infty$ -close to  $F$ .*

The most important reason why we adopt convex surfaces is a flexibility. In order to describe the property, we need the following notions. Let  $\Sigma$  be a convex surface in a contact 3-manifold  $(M, \xi)$ , and  $X$  a contact vector field defined near  $\Sigma$  which is transverse to  $\Sigma$ . The *dividing set*  $\Gamma_\Sigma$  of  $\Sigma$  is defined as

$$\Gamma_\Sigma := \{x \in \Sigma \mid \xi_x \ni X_x\}.$$

In other words, it is a set of points where the contact plane gets “vertical” to  $\Sigma$ . Note that  $\Gamma_\Sigma$  does not depend on the choice of  $X$  up to isotopy. It is known that a dividing set is a curves transverse to leaves of a characteristic foliation ([Gil]). On the other hand, a 1-dimensional foliation  $\mathcal{F}$  on  $\Sigma$  with singularity is said to be *divided* by  $\Gamma_\Sigma$  if

- $\Sigma \setminus \Gamma_\Sigma = U_+ \sqcup U_-$ ,
- $\Gamma_\Sigma$  is transverse to any leaf of  $\mathcal{F}$ ,
- there exists a vector field  $v$  which is tangent to  $\mathcal{F}$  and a volume form  $\omega$  on  $\Sigma$  which satisfy:
  - (1)  $\text{div}_\omega X > 0$  on  $U_+$ ,  $\text{div}_\omega X < 0$  on  $U_-$ ,
  - (2) the vector field  $v$  looks outward of  $U_+$  at  $\Gamma_\Sigma$ .

The following theorem makes dividing sets more flexible than characteristic foliations.

**Theorem 2.6** (Giroux, [Gil]). *Assume that  $\Sigma$  is a convex surface in a contact 3-manifold  $(M, \xi)$  with a contact vector field  $X$  transverse to  $\Sigma$ , and that  $\Gamma_\Sigma \subset \Sigma$  is a dividing set. Let  $\mathcal{F}$  be a 1-dimensional foliation with singularity on  $\Sigma$  divided by  $\Gamma_\Sigma$ . Then there exists a family  $\varphi_t: \Sigma \rightarrow M$ ,  $t \in [0, 1]$ , of embeddings which satisfies:*

- $\varphi_0 = \text{id}_\Sigma$ ,  $\varphi_t|_{\Gamma_\Sigma} = \text{id}_{\Gamma_\Sigma}$ , for any  $t \in [0, 1]$ ,
- $\varphi_t(\Sigma)$  is transverse to  $X$  for any  $t \in [0, 1]$ ,
- $(\varphi_1(\Sigma))_\xi = (\varphi_1)_*\mathcal{F}$ .

This Theorem 2.6 with Theorem 2.5 implies that it is sufficient to check dividing sets on the boundaries and their orientations in order to glue two contact manifolds.

There is a remarkable method to configure dividing curves developed by Honda [Ho]. A notion called bypass is introduced as follows. Let  $\Sigma$  be a convex surface in a contact 3-manifold,  $\Gamma_\Sigma \subset \Sigma$  a dividing set, and  $\alpha \subset \Sigma$  a Legendrian arc that intersects  $\Gamma_\Sigma$  transversely in three points  $p_1, p_2, p_3$ , where  $p_1, p_3$  are end points of  $\alpha$ . A *Legendrian curve* is a curve in a contact 3-manifold which is everywhere tangent to the contact plane field. A convex half-disk  $D$  with Legendrian boundary is called a *bypass* for  $\Sigma$  along  $\alpha$  if it satisfies the following conditions (see Figure 2.1):

- $D$  intersects  $\Sigma$  along  $\alpha$  transversely on its boundary  $\partial D$ ,
- $\text{tb}(\partial D) = -1$ .

The Thurston-Bennequin invariant  $\text{tb}(\gamma)$  of a closed Legendrian curve  $\gamma$  is a twisting number of the contact plane field along the curve with respect to a certain framing. In this case the framing depends on the disk  $D$ . The standard characteristic foliation on

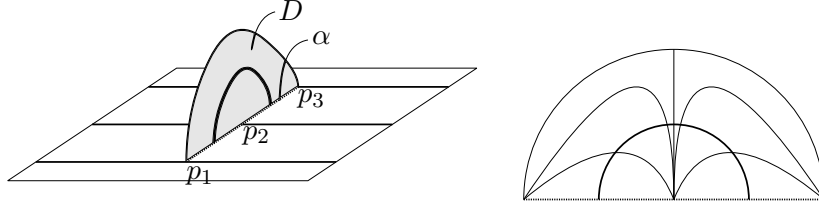


FIGURE 2.1. bypass

a bypass half-disk appears as in Figure 2.1, where the thick curve is the dividing curve. By attaching a bypass, dividing set is configured as follows.

**Lemma 2.7** (Honda, [Ho]). *Let  $\Sigma$  be a convex surface in a contact 3-manifold. Assume that there exists a bypass  $D$  for  $\Sigma$  along a Legendrian arc  $\alpha$ . Then there exists a neighborhood of  $\Sigma \cup D$  diffeomorphic to  $\Sigma \times [0, 1]$  which satisfies the following conditions:*

- $\Sigma = \Sigma \times \{\varepsilon\}$  for some  $\varepsilon \in (0, 1)$ ,
- $\Sigma \times [0, \varepsilon]$  is invariant, that is, defined by a contact vector field transverse to  $\Sigma$ ,
- $\Sigma \times \{1\}$  is convex,
- the dividing curve  $\Gamma_{\Sigma \times \{1\}}$  is obtained from  $\Gamma_{\Sigma}$  by the operation in Figure 2.2 in a neighborhood of  $\alpha$ .

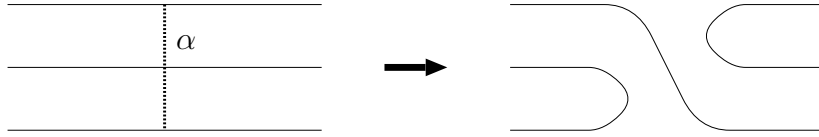


FIGURE 2.2. attaching a bypass

Finding an embedded bypass, we obtain a perturbation of a surface that causes the change of dividing curves by an attachment of the bypass like Figure 2.2. In general it is not easy to find a suitable embedded bypass. When a contact structure is overtwisted, we can find a bypass we need.

**Proposition 2.8** (Huang, [Hu]). *Let  $\Sigma$  be a convex surface in a contact 3-manifold  $(M, \xi)$ . Assume that  $\xi$  is overtwisted on  $M \setminus \Sigma$ . Then, for any Legendrian arc  $\alpha$  on  $\Sigma$  as the definition of the bypass, there exist bypasses along  $\alpha$  in  $M \setminus \Sigma$  attached from the both sides of  $\Sigma$ .*

A criterion for overtwistedness from dividing curves is introduced by Giroux.

**Proposition 2.9** (Giroux, [Gi4]). *Let  $\Sigma$  be a closed convex surface in a contact 3-manifold  $(M, \xi)$ . Unless  $\Sigma = S^2$  and the dividing set  $\Gamma_{\Sigma}$  on  $\Sigma$  is connected, the invariant tubular neighborhood of  $\Sigma$  is overtwisted if  $\Gamma_{\Sigma}$  contains a circle that is contractible in  $\Sigma$ .*

### 3. ROUND HANDLE THEORY

Round handle theory is introduced by Asimov [As] to study the Morse-Smale flow. In this paper, we need an application to 3-manifold theory by himself.

Round handle and round handle decomposition are defined as follows. Let  $M$  be a manifold of dimension  $n$  with boundary  $\partial M \neq \emptyset$ .

**Definition.** A *round handle* of dimension  $n$  and index  $k$  attached to  $M$  is defined as a pair

$$R_k = (D^k \times D^{n-k-1} \times S^1, f)$$

of a product of an  $(n-1)$ -dimensional disk  $D^k \times D^{n-k-1}$  with a circle and an attaching embedding  $f: \partial_-(D^k \times D^{n-k-1} \times S^1) \rightarrow \partial M$ , where  $\partial_-(D^k \times D^{n-k-1} \times S^1) := \partial D^k \times D^{n-k-1} \times S^1$  is the attaching region. Let  $M \cup_f R_k$  or  $M + R_k$  denote the manifold obtained from  $M$  and  $D^k \times D^{n-k-1} \times S^1$  by the attaching mapping  $f$ .

Sometimes,  $R_k$  also denotes  $D^k \times D^{n-k-1} \times S^1$  itself or the corresponding subset in  $M \cup_f R_k$ . A manifold  $M$  is said to have a *round handle decomposition* if it is obtained from  $N \times I$  by attaching round handles:

$$M = (N \times I) + R_0^1 + \cdots + R_0^{i_0} + \cdots + R_{n-1}^{i_{n-1}},$$

where  $N$  is an  $(n-1)$ -dimensional manifold without boundary and  $R_k^i$  are round handles of index  $k$ .

Round handles are used to study flow manifolds. A flow manifold is defined as follows. Let  $(M, \partial_- M)$  be a pair of a manifold  $M$  with a specific union  $\partial_- M$  of connected components of the boundary  $\partial M$ . The pair  $(M, \partial_- M)$  is called a *flow manifold* if there exists a non-singular vector field on  $M$  which looks inward on  $\partial_- M$  and outward on  $\partial_+ M := \partial M \setminus \partial_- M$ . The following property of flow manifolds is proved by Asimov.

**Theorem 3.1** (Asimov, [As]). *Let  $(M, \partial_- M)$  be a compact flow manifold whose dimension is greater than 3. Then,  $M$  has a round handle decomposition.*

By defining round surgery, the result above is applied to the study of 3-dimensional manifolds. The surgery is defined by using round handles in stead of ordinary handles. In other words, a round surgery corresponds to attaching a round handle to a cobordism. Let  $M$  be a manifold of dimension  $n$ . A *round surgery* of index  $k$  is defined as the operation removing an embedded  $\text{int}(\partial D^k \times D^{n-k} \times S^1)$  from  $M$  and regluing  $D^k \times \partial D^{n-k} \times S^1$  by the identity mapping of  $\partial D^k \times \partial D^{n-k} \times S^1$ . Applying Theorem 3.1 to a cobordism between two 3-dimensional manifolds, Asimov proved the following theorem:

**Theorem 3.2** (Asimov, [As]). *Let  $M$  be a connected closed orientable manifold of dimension 3. Then  $M$  can be obtained from a 3-dimensional sphere  $S^3$  by a finite sequence of round surgeries of index 1 and index 2.*

In the 3-dimensional case, round surgeries of index 1 and index 2 are explicitly described as follows. A round surgery of index 1 is the operation removing two open solid tori  $\text{int}(\partial D^1 \times D^2 \times S^1) = \{\text{two points}\} \times (\text{int } D^2 \times S^1)$  from a 3-manifold  $M$  and regluing a thickened torus  $D^1 \times \partial D^2 \times S^1 = I \times T^2$  by the identity mapping of a pair of tori  $\partial D^1 \times \partial D^2 \times S^1 = \{\text{two points}\} \times T^2$ . A round surgery of index 2 is the operation removing an open thickened torus  $\text{int}(D^1 \times \partial D^2 \times S^1) = \text{int}(I \times T^2)$  from a 3-manifold  $M$  and regluing two solid tori  $\partial D^1 \times D^2 \times S^1 = \{\text{two points}\} \times (D^2 \times S^1)$  by the identity mapping of a pair of tori  $\partial D^1 \times \partial D^2 \times S^1 = \{\text{two points}\} \times T^2$ .

#### 4. CONTACT ROUND SURGERY

Contact round surgeries of a contact 3-manifold of index 1 and index 2 are defined in this section. They are defined independently.

**4.1. Contact round surgery of index 1.** We define contact round surgery of index 1. As is mentioned above, a round surgery of index 1 is operated on two open solid tori in the given 3-manifold. An open solid torus can be regarded as a tubular neighborhood of a knot. In the case of a contact round surgery, it is operated along a transverse link with two components. Let  $(M, \xi)$  be a contact 3-manifold, and  $\Gamma = \gamma_1 \sqcup \gamma_2 \subset (M, \xi)$  a transverse link, where  $\gamma_1, \gamma_2$  are two connected components.

First, we determine the solid tori to remove. Since each  $\gamma_i$  is a transverse knot, we can apply the local triviality theorem. According to Theorem 2.1, there exist contact embeddings  $\varphi_i: (S^1 \times D(\varepsilon_i), \xi_0) \rightarrow (M, \xi)$  which map  $S^1 \times \{0\}$  to  $\gamma_i$  respectively, where  $D(\varepsilon)$  is a disk with radius  $\varepsilon$ . We may assume  $\text{Im } \varphi_1 \cap \text{Im } \varphi_2 = \emptyset$  by taking  $\varepsilon_i$  sufficiently small. We should be careful with the characteristic foliation  $(S^1 \times \partial D(\varepsilon_i))_{\xi_0}$  on the boundary tori. It is linear (pre-Lagrangian) with slope  $-\varepsilon^2$  with respect to the meridian  $\{\varphi = 0\}$  and the longitude  $\{x = \varepsilon, y = 0\}$  of the solid torus. Perturbing the torus slightly, we obtain a convex torus with even number of parallel dividing curves by Theorem 2.6 (see Figure 4.1). Let  $\tilde{\varphi}_i: (S^1 \times \tilde{D}(\varepsilon_i), \xi_0) \rightarrow (M, \xi)$  denote the perturbed

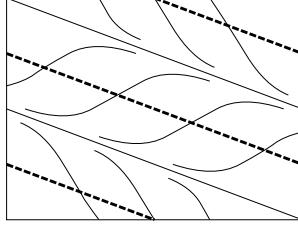


FIGURE 4.1. parallel dividing curves

ones.

Then remove  $\text{int}(\text{Im } \tilde{\varphi}_1)$  and  $\text{int}(\text{Im } \tilde{\varphi}_2)$  from the given contact 3-manifold  $(M, \xi)$ , each of which has even number of dividing curves.

Next, we construct a contact structure on the thickened torus  $D^1 \times \partial D^2 \times S^1 = I \times T^2$  which is suitable to be glued with  $M \setminus \{\text{int}(\text{Im } \tilde{\varphi}_1) \sqcup \text{int}(\text{Im } \tilde{\varphi}_2)\}$ . Let  $\xi_0 = \ker(d\varphi - ydx + xdy) = \ker(d\varphi + r^2 d\theta)$  be the standard contact structure on  $S^1 \times \mathbb{R}^2$ . Then the characteristic foliation on the torus  $T := \{(\varphi, r, \theta) \in S^1 \times \mathbb{R}^2 \mid r = 1\}$  is linear with slope  $-1$  with respect to the meridian  $\{\varphi = 0\}$  and the longitude  $\{\theta = 0\}$ . Perturbing  $T$  slightly, we have a convex torus  $\tilde{T}$  with even number of parallel dividing curves. Since  $\tilde{T}$  is convex, we have a vertically invariant tubular neighborhood  $\mathbb{T} \cong T^2 \times [-1, 1]$  of  $\tilde{T}$ , by using a contact vector field transverse to  $\tilde{T}$ . Note that the both boundary tori  $\partial \mathbb{T}$  corresponding to  $T^2 \times \{-1, 1\}$  have the same dividing sets as  $\tilde{T}$ , that is, even number of parallel dividing curves non-contractible.

Then, according to Proposition 2.4 and Theorem 2.6, the thickened torus  $(\mathbb{T}, \xi_0|_{\mathbb{T}})$  can be glued to  $(M \setminus \{\text{int}(\text{Im } \tilde{\varphi}_1) \sqcup \text{int}(\text{Im } \tilde{\varphi}_2)\}, \xi)$  according to the dividing curves. Thus we have constructed a contact round surgery of index 1 along a transverse link with two components.

Last of all, we should remark on framings of surgeries. In other words, the choices of coordinates or longitudes of the boundaries of the standard tubular neighborhood of transverse knots. The resulting manifold of a round surgery depends relatively on both framings of two knots. We can realize any round surgery as a contact surgery above. In fact, the slope of the characteristic foliation on the boundary of the standard tubular neighborhood of a transverse knot can be taken arbitrary close to 0 for any framing,



by taking the tubular neighborhood sufficiently close to the transverse knot. Then we can take any relative framings for surgeries.

**4.2. Contact round surgery of index 2.** We define contact round surgery of index 2. First of all, we should remark that this surgery is not always defined. As is mentioned in the last part of Section 3, a round surgery of index 2 is operated along a 2-dimensional torus embedded in the given 3-manifold. An open thickened torus as a tubular neighborhood of the embedded torus is removed. And then two solid tori are reglued. In the case of a contact round surgery of a contact 3-manifold  $(M, \xi)$ , it is operated along an embedded torus  $T \subset (M, \xi)$ . We impose this torus the condition that

$$(4.1) \quad \langle e(\xi), [T] \rangle = 0,$$

where  $e(\xi) \in H^2(M; \mathbb{Z})$  is the Euler class of the contact structure, and  $[T] \in H_2(M; \mathbb{Z})$  is the fundamental class. This condition is translated to Equation (4.2) in terms of convex surface theory if  $T$  is a convex torus. In the first part of this subsection, we discuss on this condition. And then, under this condition, we define contact round surgery of index 2.

**4.2.1. The condition on the embedded tori.** Now, we discuss the meaning of the condition on an embedded torus  $T \subset (M, \xi)$  where a contact round surgery is operated. The condition is required in the construction of a contact structure on the surgered manifold. In the following construction, we remove a tubular neighborhood of  $T \subset (M, \xi)$ , and reglue two contact solid tori. In order to do that, we need contact structures on a solid torus which have the same characteristic foliation (or dividing set) as  $T \subset (M, \xi)$ . In other words, we need to extend the contact structure determined by the characteristic foliation to the whole solid torus. The Euler class  $e(\xi) \in H^2(M; \mathbb{Z})$  of  $\xi$  evaluated with  $[T] \in H_2(M; \mathbb{Z})$  is an obstruction to the extension as a plane field by the obstruction theory (see for example [Br]). That is, Condition (4.1) guarantees the extension as a plane field. Actually, contact structure along  $T$  extends as a contact structure in this case. It is proved in the construction below.

Condition (4.1) for a 2-dimensional torus  $T$  embedded in a contact 3-manifold  $(M, \xi)$  is natural in some cases. We have the following two examples.

**Example 1.** If  $\xi$  is a tight contact structure on a 3-manifold  $M$ , any embedded torus  $T \subset (M, \xi)$  satisfies Condition (4.1). In fact, for tight contact structures, the following property is known:

**Theorem 4.1** (Eliashberg, [El3]). *Let  $(M, \xi)$  be a tight contact 3-manifold. For any closed orientable surface  $\Sigma$  embedded in  $(M, \xi)$ , the following inequality holds:*

$$|\langle e(\xi), [\Sigma] \rangle| \leq \begin{cases} 0 & \text{if } \Sigma = S^2, \\ -\chi(\Sigma) & \text{otherwise,} \end{cases}$$

where  $\chi(\Sigma)$  is the Euler characteristic of  $\Sigma$ .

In the case that we consider now,  $\Sigma$  is a torus  $T$  and then  $\chi(T) = 0$ . Therefore,  $\langle e(\xi), [T] \rangle = 0$ .  $\square$

**Example 2.** If  $M$  is a closed orientable 3-manifold, any embedded torus  $T \subset (M, \xi)$  which separates the manifold  $M$  satisfies Condition (4.1). In fact,  $\langle e(\xi), [T] \rangle$  is an obstruction to extending the contact plane field  $\xi|_T$  along  $T$  to the manifold bounded

by  $T$  as a plane field. In this case,  $\xi$  itself is the extension. Then  $\langle e(\xi), [T] \rangle$  should vanish.  $\square$

The case when the contact structure  $\xi$  on a closed 3-manifold  $M$  is overtwisted and the embedded torus  $T \subset M$  does not separate the underlying manifold is left. In that case, we can not define contact round surgery of index 2 operated along the torus, which has  $\langle e(\xi), [T] \rangle \neq 0$ . However, after modifying the contact structure  $\xi$  suitably to  $\xi'$  on  $M$  with  $\langle e(\xi'), [T] \rangle = 0$ , we can operate a contact round surgery of index 2 along the same  $T$ . we observe these operation in Section 6.

**4.2.2. Definition of a contact round surgery of index 2.** Now, we define contact round surgery of index 2 under Condition (4.1). Let  $(M, \xi)$  be a contact 3-manifold, and  $T \subset (M, \xi)$  an embedded 2-dimensional torus along which the round surgery is operated. Assume that  $T$  satisfies Condition (4.1). We may also assume that  $T \subset (M, \xi)$  is convex due to Theorem 2.5. In addition, we take a meridian  $\hat{\mu} \subset T$ , or  $\hat{\mu} \in H_1(T, \mathbb{Z})$ . It corresponds to a meridian  $\partial D^2 \times \{*\} \times \{*\} \subset \partial D^2 \times D^1 \times S^1$  of the thickened torus to be removed. In other words, it will be the meridians  $\partial D^2 \times \partial D^1 \times \{*\} \subset D^2 \times \partial D^1 \times S^1$  of the solid tori to be reglued. In the following, we call it the *surgery meridian*.

First, we perturb the embedded torus  $T$  to a suitable position. We isotope  $T$  so that the dividing set  $\Gamma_T$  on  $T$  consists of homotopically non-trivial parallel curves. In order to do that, we should check possible dividing sets on a convex 2-dimensional torus under Condition (4.1). According to the properties of dividing sets, a dividing set on a convex 2-dimensional torus consists of even-number of parallel homotopically non-trivial curves and some homotopically trivial curves who never intersect each other. Further, on the Euler characteristics of the domains  $U_{\pm} \subset T^2$  divided by dividing sets, there exists the following constraint:

$$(4.2) \quad \chi(U_+) = \chi(U_-) = 0.$$

In fact, we have the following formulas for a convex closed surface  $\Sigma$  in a contact 3-manifold  $(N, \tilde{\xi})$  with positive and negative regions  $R_{\pm}$  (see [Ho]):

$$\chi(\Sigma) = \chi(R_+) + \chi(R_-), \quad \langle e(\tilde{\xi}), [\Sigma] \rangle = \chi(R_+) - \chi(R_-).$$

In this case,  $\chi(\Sigma) = \chi(T) = 0$  since  $T$  is a torus, and  $\langle e(\xi), [T] \rangle = 0$  from the assumption. Then we have  $\chi(U_+) = \chi(U_-) = 0$ .

Then we consider removing homotopically trivial dividing curves in what follows. The method to be applied is the bypass attachment. Now that the dividing set  $\Gamma_T$  on  $T$  has a homotopically trivial dividing curves, on the transversely invariant tubular neighborhood  $U \subset M$  of  $T$ , the contact structure  $\xi$  is overtwisted from the Giroux criterion (Proposition 2.9). Further,  $\xi$  is overtwisted on  $U \setminus T$ . Then, by Proposition 2.8, we can find any embedded bypass we need. Isotoping  $T$  along the bypass, the dividing set  $\Gamma_T$  is modified in the same manner as the bypass attachment (See Lemma 2.7, Figure 2.2). Therefore, we have only to follow such modifications of the dividing set in order to cancel homotopically trivial dividing curves.

We introduce basic operations creating or canceling a pair of homotopically trivial dividing curves. By attaching a bypass, we can create or cancel a pair of homotopically trivial dividing curves. See Figure 4.2 for an independent pair and Figure 4.3 for a nested pair.

By using Operations I, II, III, and IV, any possible dividing set is reduced to homotopically non-trivial dividing curves. First, we apply Operation IV to cancel nested pair

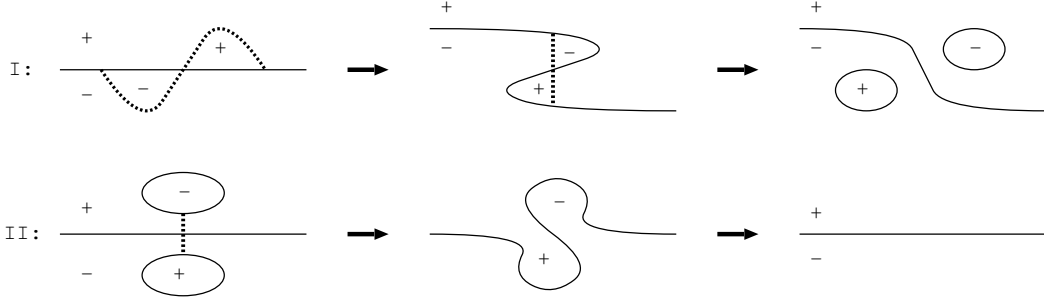


FIGURE 4.2. creating and canceling independent contractible dividing curves.

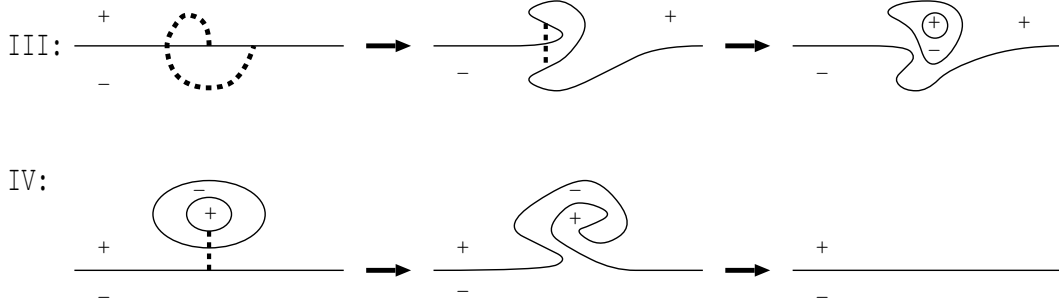


FIGURE 4.3. nested pair of dividing curves

of homotopically trivial dividing curves. Then the rest of homotopically trivial dividing curves are single independent ones. Such a single homotopically trivial dividing curve is moved to other domain by the combination of Operations I and II (see Figure 4.4). All homotopically trivial dividing curves which bound positive domain are gathered

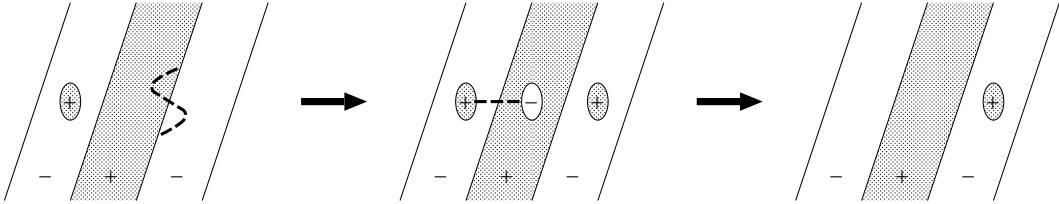


FIGURE 4.4. move to other domain

in one negative strip, and all those which bound negative domain are gathered in a positive strip next to the positive strip above. From Condition (4.2), the numbers of homotopically trivial dividing curves bounding positive or negative domains must be the same. Therefore, all of them are canceled by Operation II. Then the given contact solid torus with convex boundary is reduced to the contact solid torus without homotopically trivial dividing curves on the boundary.

Now, we operate a round surgery of index 2 along the isotoped convex torus  $T$  with no homotopically trivial dividing curve. Recall that the dividing set  $\Gamma_T$  on  $T$  consists of even number of parallel dividing curves.

First, we determine the thickened torus to remove. Since  $T \subset (M, \xi)$  is convex, there exists a contact vector field  $X$  transverse to  $T$  on a neighborhood of  $T \subset M$ . By using the flow of the vector field  $X$ , an open tubular neighborhood  $\varphi: T \times (-\varepsilon, \varepsilon) \rightarrow (M, \xi)$  is

constructed, where  $\varphi$  maps  $T \times \{0\}$  to  $T \subset M$  identically. Remove  $\text{Im } \varphi \subset M$  from the given contact 3-manifold  $(M, \xi)$ . We should be careful with the characteristic foliation on the boundary of  $M \setminus \text{Im } \varphi$ . It is diffeomorphic to the two copies of the characteristic foliation  $T_\xi$  because the vector field  $X$  is contact.

Next, we need two contact solid tori to reglue. Recall that we have a surgery meridian  $\hat{\mu}$  of  $T$ . For one of the contact solid torus it should be the meridian. On the other hand, for the other one,  $-\hat{\mu}$  should be the meridian. In other words, the characteristic foliations on the boundary tori are the mirror image of each other. The model is constructed as follows. Let  $\zeta$  be an overtwisted contact structure  $\{(\cos r^2)d\varphi + r(\sin r^2)d\theta = 0\}$  on  $S^1 \times \mathbb{R}^2$  with cylindrical coordinates  $(\varphi, r, \theta)$  as in Section 4.1. As is seen above, the characteristic foliation  $(S^1 \times \partial D(\rho))_\zeta$  is a linear foliation with slope  $-\tan \rho^2$ . By perturbing it, we have even number of parallel dividing curves by Theorem 2.6 (see Figure 4.1). For  $0 < \rho < \pi$ , we obtain any non-zero slope. Even in the case of meridional dividing curves, we obtain them by perturbing  $S^1 \times \partial D(\sqrt{\pi})$ . Thus we obtain any required homotopically non-trivial dividing curves by perturbing  $S^1 \times \partial D(\rho) \subset (S^1 \times \mathbb{R}^2, \zeta)$ ,  $0 < \rho \leq \sqrt{\pi}$ . Then, if the slope is not 0, the model is considered as a part of a contact 3-manifold  $(S^1 \times S^2, \zeta_0)$  which is a union of the two copies of  $(S^1 \times D(\sqrt{\pi/2}), \zeta)$ . Even if the slope is 0, it is considered as a part of  $(S^1 \times S^2, \zeta_1)$  which is from the two copies of  $(S^1 \times D(\sqrt{\pi}), \zeta)$ . Therefore, the closure of the complement of the obtained solid torus is the other required one.

We glue the obtained contact solid tori to  $(M \setminus \text{Im } \varphi, \zeta)$  as follows. From the construction, they have the same dividing curves. By Theorem 2.6, we may regard that they have the same characteristic foliations. Then, by Proposition 2.4, we can glue them.

Last of all, we remark that we can define contact round surgery for any framing, or surgery meridian, of the embedded torus because we can arrange the slope of the dividing curves of the model of contact solid torus for that.

## 5. LUTZ TWIST AND GIROUX TORSION

There exist two important notions on contact structures on 3-manifolds, the so called Lutz twist and Giroux torsion. First, we recall the definitions of them. Then we show that the Lutz twist is realized by contact round surgeries. Further, we discuss some relation to the Giroux torsion.

**5.1. Definitions.** In this subsection, we review the definitions of the Lutz twists and the Giroux torsion.

**5.1.1. Lutz twist along  $S^1$ .** Let us begin by reviewing the definition of the original Lutz twist. It is an operation modifying a contact structure along a transverse knot. Let  $\Gamma$  be a transverse knot in a contact 3-manifold  $(M, \xi)$ . There exists a tubular neighborhood of  $\Gamma$  which is contactomorphic to  $(S^1 \times D(\rho), \xi_0) =: U$  for some small radius  $\rho > 0$ , by Theorem 2.1. In order to define the Lutz twist, we need the standard overtwisted contact structure. Let  $\zeta$  be an overtwisted contact structure on  $S^1 \times \mathbb{R}^2$  defined as

$$\zeta := \ker\{(\cos r^2)d\varphi + (\sin r^2)d\theta\},$$

where  $(\varphi, r, \theta) \in S^1 \times \mathbb{R}^2$  are the cylindrical coordinates. Note that  $(S^1 \times \mathbb{R}^2, \xi_0)$  is isotopic to  $(S^1 \times \text{int } D(\pi/2), \zeta)$ . By this correspondence, the tubular neighborhood  $U = (S^1 \times D(\rho), \xi_0)$  is identified with  $(S^1 \times D(\bar{\rho}), \zeta) =: \bar{U}$  for  $\bar{\rho} > 0$  satisfying  $\rho^2 = \tan \bar{\rho}^2$ ,

$0 < \bar{\rho}^2 < \pi/2$ . The *simple Lutz twist* is the operation replacing  $U \cong \bar{U} = (S^1 \times D(\bar{\rho}), \zeta)$  with  $(S^1 \times D(\sqrt{\bar{\rho}^2 + \pi}), \zeta) =: \tilde{U}_\pi$  (see Figure 5.1-(I)). In fact, since the characteristic

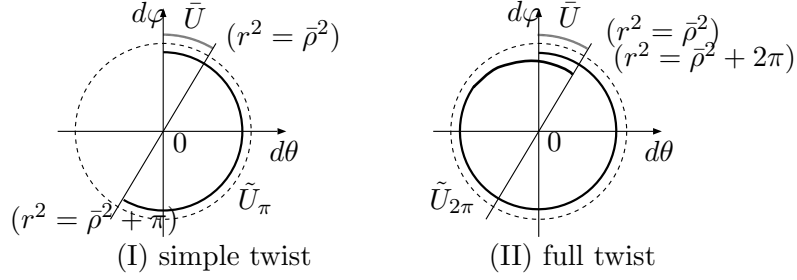


FIGURE 5.1. Lutz twist along  $S^1$   
(situations of covectors or contact form  $\zeta$ )

foliations  $(\partial \bar{V})_\zeta = (S^1 \times \partial D(\bar{\rho}))_\zeta$  and  $(\partial \tilde{V}_\pi)_\zeta = (S^1 \times \partial D(\sqrt{\bar{\rho}^2 + \pi}))_\zeta$  are the same, these two contact solid tori can be replaced. Note that the contact plane of the second one twists a half time, or  $\pi$ , more than the first one along a radial  $r$ -axis. Similarly, the *full Lutz twist* is defined as the operation replacing  $V = (S^1 \times D(\rho), \xi_0)$  with  $\tilde{V}_{2\pi} := (S^1 \times D(\sqrt{\bar{\rho}^2 + 2\pi}), \zeta)$  (see Figure 5.1-(II)). In this case, the contact plane twists one time, or  $2\pi$ , more. We should remark here that a simple Lutz twist does change the homotopy class of the contact structure as plane fields, a full Lutz twist does not though. It is clear that a Lutz twist makes a contact structure be overtwisted.

**5.1.2. Lutz twist along  $T^2$ .** Next, we introduce the Lutz twist along a certain torus. Let  $T$  be a torus embedded in a contact 3-manifold  $(M, \xi)$ . Assume that it is pre-Lagrangian. An embedded torus  $T \subset (M, \xi)$  is said to be *pre-Lagrangian* if the characteristic foliation  $T_\xi$  is linear with closed leaves. In other words,  $T$  is foliated by Legendrian circles. Then it is well known (see [Ge], [Gi3]) that it has a tubular neighborhood which is contactomorphic to a tubular neighborhood of the 2-dimensional torus  $\{(\varphi, r, \theta) \in S^1 \times \mathbb{R}^2 \mid r = \rho\} \subset (S^1 \times \mathbb{R}^2, \xi_0)$ , for some radius  $\rho > 0$ . We may regard the tubular neighborhood as

$$\begin{aligned} V &:= (\{(\varphi, r, \theta) \mid \rho - \delta_1 < r < \rho + \delta_2\}, \xi_0) \\ &\cong \bar{V} := (\{(\varphi, r, \theta) \mid \bar{\rho}^2 - \delta < r^2 < \bar{\rho}^2 + \delta\}, \zeta), \end{aligned}$$

where  $\rho^2 = \tan \bar{\rho}^2$ ,  $(\rho - \delta_1)^2 = \tan(\bar{\rho}^2 - \delta)$ ,  $(\rho + \delta_2)^2 = \tan(\bar{\rho}^2 + \delta)$ . The *simple Lutz twist* (or  $\pi$ -Lutz twist) along  $T \subset (M, \xi)$  is defined as the operation replacing the tubular neighborhood  $\bar{V}$  of  $T$  with

$$\tilde{V}_\pi := (\{(\varphi, r, \theta) \mid \bar{\rho}^2 - \delta < r^2 < \bar{\rho}^2 + \pi + \delta\}, \zeta),$$

(see Figure 5.2). We can define the *full Lutz twist* (or  $2\pi$ -Lutz twist) along  $T$  by using  $\tilde{V}_{2\pi} := \{(\varphi, r, \theta) \mid \bar{\rho}^2 - \delta < r^2 < \bar{\rho}^2 + 2\pi + \delta\}$ .

The Lutz twist along a pre-Lagrangian torus is considered as a generalization of that along a transverse knot. In fact, for a transverse knot  $\Gamma \subset (M, \xi)$ , there exists a tubular neighborhood  $U \subset (M, \xi)$  of  $\Gamma$  which is contactomorphic to a tubular neighborhood of  $S^1 \times \{0\} \subset (S^1 \times \mathbb{R}^2, \xi_0)$ . Then, in the tubular neighborhood  $U$ , we have a pre-Lagrangian torus  $T \subset U$  corresponding to some torus  $\{(\varphi, r, \theta) \mid r = \rho\} \subset (S^1 \times \mathbb{R}^2, \xi_0)$ . The Lutz twist along  $T$  is equivalent to that along  $\Gamma$ .

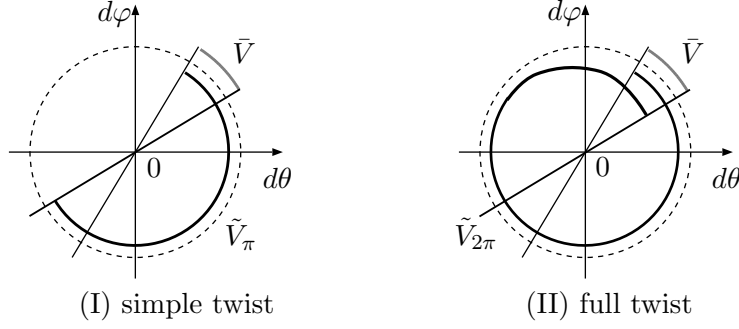


FIGURE 5.2. Lutz twist along a torus

We should mention that the Lutz twist along a pre-Lagrangian torus may not create any overtwisted disc. However, the Lutz twist along a pre-Lagrangian torus creates the following important thing.

**5.1.3. Giroux torsion.** We review the definition of the Giroux torsion. A contact manifold  $(M, \xi)$  is said to have the *Giroux torsion at least*  $n \in \mathbb{N}$  if there exists a contact embedding  $f_n: (T^2 \times I, \tilde{\zeta}_n) \rightarrow (M, \xi)$ , where  $\tilde{\zeta}_n$  is a contact structure on  $T^2 \times I$  with coordinates  $(\varphi, r, \theta) \in S^1 \times I \times S^1 \subset S^1 \times \mathbb{R}^2$  defined as

$$\tilde{\zeta}_n := \ker\{\cos(2n\pi r)d\theta + \sin(2n\pi r)d\varphi\}.$$

The supremum of these numbers for all such embeddings to the contact manifold  $(M, \xi)$  is called the *Giroux torsion* of  $(M, \xi)$ . Let  $\text{Tor}(M, \xi)$  denotes it. If there exists no such embedding,  $(M, \xi)$  is said to have Giroux torsion 0. The definition can be extended for half-integers  $n = m/2$ ,  $m \in \mathbb{N}$ . Especially, for  $n = 1/2$ , we call the contact embedding  $f_{1/2}: (T^2 \times I, \tilde{\zeta}_n) \rightarrow (M, \xi)$  a *half Giroux torsion unit*.

The Giroux torsion is an invariant of contact 3-manifold introduced by Giroux [Gi2], which explicitly appears in the classification of tight contact structures on the 3-dimensional torus  $T^3$  due to Giroux and Kanda [K] independently. It is proved that a closed contact manifold is not strongly symplectically fillable if it has the Giroux torsion greater than 0 (see [Ga]).

The Giroux torsion is closely related to the Lutz twist. The Lutz twist along a torus make a Giroux torsion unit. In fact, comparing the contact structures  $\tilde{\zeta}_n$  and  $\zeta$ , we obtain that the substitute thickened torus in the Lutz twist includes a half Giroux torsion unit.

**5.2. Realization by round surgeries.** Now, we show that the Lutz twist is realized by a certain 4-tuple of contact round surgeries. More precisely, we construct two pairs of contact round surgeries of index 1 and index 2. By such contact round surgeries, we will realize the Lutz twist along a pre-Lagrangian torus. As is mentioned in Subsubsection 5.1.2, this realizes the Lutz twist along a transverse knot as well. The claim in this subsection is the following.

**Theorem 5.1.** *A simple Lutz twist is realized by a certain ordered 4-tuple of a contact round surgeries of index 1 and those of index 2.*

First of all, we confirm the starting situations locally. Let  $T \subset (M, \xi)$  be a pre-Lagrangian torus in a contact 3-manifold  $(M, \xi)$  where we will operate the Lutz twist. Along the pre-Lagrangian torus  $T$ , there exist the standard tubular neighborhood  $U \subset$

$(M, \xi)$  which is contactomorphic to a tubular neighborhood of the torus  $\{(\varphi, r, \theta) \mid r = \rho\} \subset (S^1 \times \mathbb{R}^2, \zeta = \ker\{(\cos r^2)d\varphi + (\sin r^2)d\theta\})$  for some  $\rho > 0$ . We may assume that the tubular neighborhood is  $U = (\{(\varphi, r, \theta) \mid \pi/4 - \varepsilon < r^2 < \pi/4 + \varepsilon\}, \zeta)$  by taking longitude and meridian suitably. In the following, we discuss by using this local model.

We operate a contact round surgery of index 2 along  $T \subset (U, \zeta)$  first. Since the torus clearly satisfies Condition (4.1), we can operate a contact round surgery of index 2 along  $T$ . In the local model  $(U, \zeta)$ , we determine the surgery meridian as follows. Let  $\mu \in H_1(T; \mathbb{Z})$ , or  $\mu \subset T$ , be the meridian of  $T$  corresponding to  $\{\varphi = 0\}$  and  $\lambda \in H_1(T; \mathbb{Z})$ , or  $\lambda \subset T$ , the longitude of  $T$  corresponding to  $\{\theta = 0\}$ . We take  $\hat{\mu} := \lambda$  as the surgery meridian. Note that, by the framing  $(\mu, \lambda)$  of  $T$ , the characteristic foliation  $T_\zeta$  on  $T$  is represented by  $\lambda - \mu$ .

Now, we operate a contact round surgery of index 2 along  $T \subset (U, \zeta)$  with surgery meridian  $\hat{\mu} = \lambda$ . Cutting  $(U, \zeta)$  open along  $T$ , we reglue two contact solid tori. These tori are prepared according to the surgery meridian  $\hat{\mu}$  and the characteristic foliation  $T_\zeta$ . Let  $N_1 \cong S^1 \times D^2$  denote a solid torus whose boundary has the same orientation as  $T$ , and  $N_2$  denote a solid torus whose boundary has the opposite orientation to  $T$ . For  $N_1$ , the meridian  $\mu_1$  of the boundary torus  $\partial N_1 \cong S^1 \times \partial D^2$  corresponds to  $\hat{\mu} = \lambda$ . As its longitude, we take  $\lambda_1 \subset \partial N_1$ , or  $\lambda_1 \in H_1(\partial N_1; \mathbb{Z})$ , that corresponds to  $-\mu \in H_1(T; \mathbb{Z})$ , so that the orientation of  $\partial N_1$  is the same as  $T$ . Then the characteristic foliation  $T_\zeta$  corresponds to the foliation on  $\partial N_1$  represented by  $\mu_1 + \lambda_1$ . Therefore,  $N_1$  should be isotopic to  $(S^1 \times D(\sqrt{3\pi/4}), \zeta)$  (see Figure 5.3-(I)). Similarly, for  $N_2$ , the meridian  $\mu_2$  of  $\partial N_2$  corresponds to  $-\hat{\mu} = -\lambda$  and a longitude  $\lambda_2$  corresponds to  $-\mu$ , so that the orientation of  $\partial N_2$  is opposite to  $T$ . Then the characteristic foliation  $T_\zeta$  corresponds to the foliation on  $\partial N_2$  represented by  $-\mu_2 + \lambda_2$ . Therefore,  $N_2$  should be isotopic to  $(S^1 \times D(\sqrt{\pi/4}), \zeta)$  (see Figure 5.3-(I)). Gluing these two solid tori  $N_1$  and  $N_2$  to  $(U \setminus T, \zeta)$ , we obtain a new contact manifold.

Next, we operate a contact round surgery of index 1. Recall that in the previous contact round surgery of index 2, we glue two contact solid tori  $N_1$  and  $N_2$ . Let  $\gamma_1 \subset N_1$  and  $\gamma_2 \subset N_2$  be transverse knots corresponding to  $S^1 \times \{0\} \subset (S^1 \times D(\sqrt{3\pi/4}), \zeta) = N_1$ ,  $(S^1 \times D(\sqrt{\pi/4}), \zeta) = N_2$ , respectively. We operate a contact round surgery of index 1 along the transverse link  $\gamma_1 \sqcup \gamma_2$ . In other words, removing tubular neighborhoods of  $\gamma_1$  and  $\gamma_2$ , we glue two boundary tori together. In order to determine the framing of surgery, we determine tubular neighborhoods of  $\gamma_1$  and  $\gamma_2$ . As the tubular neighborhoods, we take  $V_1 := (S^1 \times D(\sqrt{\pi/4}), \zeta) \subset N_1$  for  $\gamma_1$  and  $V_2 := (S^1 \times D(\sqrt{\pi/4}), \zeta) \subset N_2$  for  $\gamma_2$  (see Figure 5.3-(II)). Then removing  $\text{int } V_1$  and  $\text{int } V_2$ , we glue two boundary tori  $\partial V_1$  and  $\partial V_2$  so that their meridians and characteristic foliations agree. Thus we obtain a new contact manifold.

As a result, these two contact round surgeries amount to the following. In total, cutting the contact manifold  $(U, \zeta)$  open along the pre-Lagrangian torus  $T \subset (U, \zeta)$ , we glued the boundary tori to the contact thickened torus  $\{(\varphi, r, \theta) \mid 0 \leq r^2 \leq \pi/2\} \cong T^2 \times I$  from the both sides. In fact, in the first contact round surgery of index 2, cutting  $(U, \zeta)$  open along  $T$ , we glued two solid tori  $N_1$  and  $N_2$ . Then, in the second contact round surgery of index 1, removing

$$\begin{aligned} \text{int } V_1 &= (S^1 \times \text{int } D(\sqrt{\pi/4}), \zeta) \subset (S^1 \times D(\sqrt{3\pi/4}), \zeta) = N_1, \\ \text{int } V_2 &= (S^1 \times \text{int } D(\sqrt{\pi/4}), \zeta) \subset (S^1 \times D(\sqrt{\pi/4}), \zeta) = N_2, \end{aligned}$$

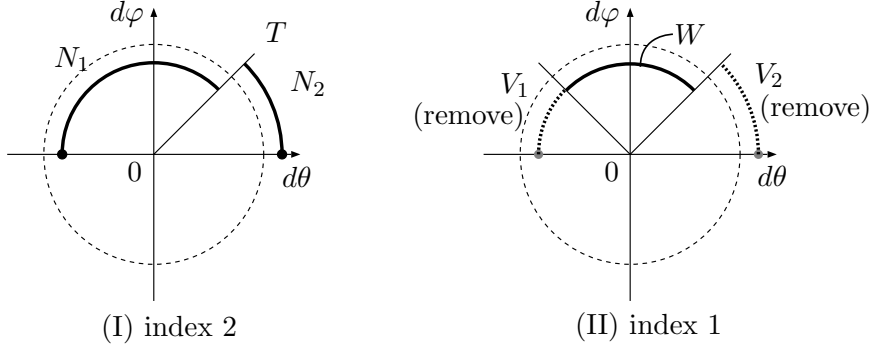


FIGURE 5.3. Contact round surgeries  
(situations of covectors or contact forms)

we glued  $\partial V_1$  and  $\partial V_2$  together. As a result, the contact thickened torus

$$W_1 := N_1 \setminus \text{int } V_1 = (\{(\varphi, r, \theta) \mid \pi/4 \leq r^2 \leq 3\pi/4\}, \zeta)$$

is left in the resulting manifold. It is isotopic to  $(\{(\varphi, r, \theta) \mid 0 \leq r^2 \leq \pi/2\}, \zeta)$  (see Figure 5.3-(II)). We should remark that, at this moment, the underlying manifold has been modified.

We operate the same pair of contact round surgeries again along the pre-Lagrangian torus  $\hat{T} = \partial V_1 = \partial V_2$ , where the first pair of round surgeries finished. Then we obtain another contact thickened torus  $W_2 := (\{(\varphi, r, \theta) \mid 0 \leq r^2 \leq \pi/2\}, \zeta)$  just next to the previous  $W_1$ .

These two pairs of contact round surgeries amount to the simple Lutz twist along  $T \subset (U, \zeta)$ . In fact, the contact thickened tori  $W_1$  and  $W_2$  are glued so that their meridians and characteristic foliations agree. Then the combined contact thickened torus  $W_1 \cup W_2$  is isotopic to  $(\{(\varphi, r, \theta) \mid 0 \leq r^2 \leq \pi\}, \zeta)$ . We should remark that the underlying manifold has recovered to the original shape because the boundary tori  $\{r^2 = 0\}$  and  $\{r^2 = \pi\}$  have the same characteristic foliation. This implies that the total operation is nothing but the simple Lutz twist.

Thus, Theorem 5.1 has been proved.

## 6. CONCLUSION

Theorem A and Theorem B are proved in this section. In the first subsection, we show that any closed orientable 3-manifold admits a contact structure, by using round surgeries. In the second subsection, we show that any contact structure on any closed orientable 3-manifold is obtained from the standard contact 3-sphere by contact round surgeries.

**6.1. Construction on any manifold.** We give a proof of Theorem A, that is, an alternative proof of the theorem firstly proved by Martinet [Mar]. In this paper, it is proved by using round surgeries. We use the method as follows. According to Theorem 3.2, any closed orientable 3-manifold is constructed from a 3-dimensional sphere by a sequence of round surgeries of index 1 and index 2. What is to be proved is that the given sequence of round surgeries can be operated as contact round surgeries. First, we show that each round surgery of index 1 can be operated as a contact round surgery. On the other hand, some round surgeries of index 2 can not be operated as contact round surgeries directly. Recall that a contact round surgery of index 2 is



defined along a torus  $T^2$  embedded into a contact 3-manifold  $(M, \xi)$  which satisfies Condition (4.1):  $\langle e(\xi), [T^2] \rangle = 0$ . We should discuss the case when the round surgery that we would like to operate as a contact round surgery is operated along a torus  $T \subset M$  with  $\langle e(\xi), [T] \rangle \neq 0$ . In that case, we modify the contact structure  $\xi$  so that it satisfies  $\langle e(\xi), [T] \rangle = 0$ . At the end of this subsection, we discuss the existence of a sequence of contact round surgeries without modifying the contact structures.

First, we deal with round surgeries of index 1. A surgery of this kind is operated along a link with two components. According to Proposition 2.2, any curve is approximated by a positively transverse curve. Then any round surgery of index 1 is operated as a contact round surgery.

The argument on round surgeries of index 2 is much more complicated. A round surgery of index 2 is operated along an embedded 2-dimensional torus. A contact round surgery of index 2 is operated along a torus  $T$  in a contact 3-manifold  $(M, \xi)$  with Condition (4.1):  $\langle e(\xi), [T] \rangle = 0$ . By Theorem 2.5, the torus  $T$  is approximated by a convex torus. Since we deal with a closed contact 3-manifold  $(M, \xi)$ , Condition (4.1) is satisfied if  $\xi$  is tight or  $T \subset M$  separates  $M$  (see Examples 1 and 2 in Section 4). The case when the contact structure  $\xi$  is overtwisted and the embedded torus  $T \subset M$  does not separate  $M$  is left to be discussed.

Now, we discuss the case when we can not operate a contact round surgery directly. Let  $\xi$  be an overtwisted contact structure on a closed 3-manifold  $M$ , and  $T \subset (M, \xi)$  an embedded torus which does not separate  $M$ . When the contact structure  $\xi$  is overtwisted, there exists a case when  $\langle e(\xi), [T] \rangle \neq 0$ . Assume that  $\langle e(\xi), [T] \rangle \neq 0$ .

First, we translate the condition in terms of the Euler characteristic of the regions divided by the dividing set. Let  $R_{\pm} \subset T$  be the positive and negative regions. From Formulas (4.2.2), we have  $\langle e(\xi), [T] \rangle = 2\chi(R_+)$  since  $\chi(\Sigma) = \chi(T^2) = 0$ . Therefore, the assumption  $\langle e(\xi), [T] \rangle \neq 0$  implies  $\chi(R_+) \neq 0$ .

The Euler characteristic  $\chi(R_+)$  of the positive region  $R_+$  can be changed by modifying the contact structure  $\xi$  as follows. Recall that the embedded torus  $T \subset (M, \xi)$  does not separate the manifold  $M$ . Then we have a transverse knot  $K \subset (M, \xi)$  which intersects the positive region  $R_+ \subset T$  once transversely. By the Lutz twist along  $K$  sufficiently close to  $K$ , we obtain a contact structure  $\bar{\xi}$  modified from  $\xi$  around  $K$ . With respect to this  $\bar{\xi}$ , we have another homotopically trivial dividing curve on  $T$  around the point where  $K$  intersects  $T$  (see Figure 6.1). Let  $\bar{R}_{\pm}$  denote the new positive and negative

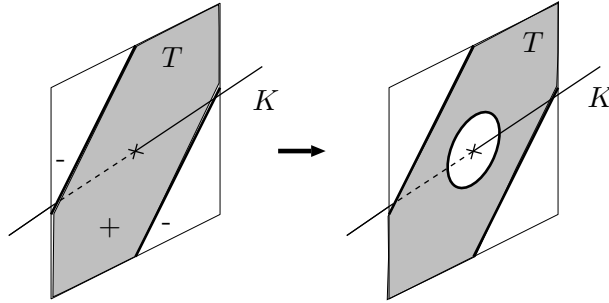


FIGURE 6.1. making a homotopically trivial dividing curve

regions on  $T$  with respect to  $\bar{\xi}$ . Then we have

$$\chi(\bar{R}_+) = \chi(R_+) - 1, \quad \chi(\bar{R}_-) = \chi(R_-) + 1.$$

Similarly, by the simple Lutz twist along a transverse knot  $K'$  which intersects  $R_-$ , the Euler characteristics are changed as

$$\chi(\bar{R}'_+) = \chi(R_+) + 1, \quad \chi(\bar{R}'_-) = \chi(R_-) - 1,$$

where  $\bar{R}'_{\pm}$  are the positive and negative regions after the simple Lutz twist along  $K'$ . Then, by applying the simple Lutz twists suitably, we obtain a contact structure  $\zeta$  on  $M$  with respect to which  $\langle e(\zeta), [T] \rangle = 0$  holds. Note that we have not changed the manifold  $M$  and the embedded torus  $T$  but a contact structure on  $M$ .

Then we can conclude that any given round surgery of index 2 can be operated as a contact round surgery of index 2, after changing the contact structure if necessary. In fact, even if the given 2 dimensional torus  $T$  in a contact 3-manifold  $(M, \xi)$  satisfies  $\langle e(\xi), [T] \rangle \neq 0$ , we have another contact structure  $\zeta$  on  $M$  with  $\langle e(\zeta), [T] \rangle = 0$ , as above. Then we can operate the given round surgery of index 2 of  $M$  along  $T \subset M$  as a contact round surgery of index 2 of  $(M, \zeta)$  along  $T$ .

This completes the proof of Theorem A.  $\square$

*Remark.* Although the argument above is sufficient to prove the existence of a contact structure on a given manifold, we remark one thing for the discussion in the following subsection. In the construction above, we use the Lutz twists other than contact round surgeries. However, it had been proved that the simple Lutz twist is described as a pair of contact round surgeries (see Theorem 5.1). Therefore, by making detours, we obtain a sequence of round surgeries of index 1 and index 2 from the given contact 3-sphere to any closed orientable 3-manifold which can be operated as direct contact round surgeries. In other words, we can construct a contact structure on any closed orientable 3-manifold from a contact 3-sphere only by contact round surgeries without changing the intermediate contact structures.

**6.2. Construction of any contact structure.** Theorem B is proved in this section. We show that any contact structure on any closed oriented 3-manifold is constructed from the standard contact 3-sphere by contact round surgeries of index 1 and index 2. It is proved in the following 2 steps. In the first step, we show that any closed contact 3-manifold is constructed from a 3-dimensional sphere with some contact structure by contact round surgeries. In the second step, we show that any contact structure on a 3-dimensional sphere is constructed from the standard contact structure by contact round surgeries.

**Step 1:** First, we verify that the procedure, topological round surgery, is reversible. Recall that a round surgery implies attaching a round handle to a cobordism. If an  $n$ -dimensional manifold  $N$  is obtained from a manifold  $M$  by a round surgery of index  $k$ , the boundary of  $W := (M \times I) + R_k$  consists of  $M$  and  $N$ . From the Poincaré duality trick, the cobordism  $W$  between  $M$  and  $N$  can be considered as  $W = (N \times I) + R_{n-k-1}$  (see [As]). This implies that  $M$  is obtained from  $N$  by a round surgery of index  $n - k - 1$ . Therefore, Theorem 3.2 implies the 3-dimensional sphere  $S^3$  is obtained from any connected closed orientable 3-manifold by round surgeries of index 1 and index 2.

Next, we show that contact round surgeries are reversible. In the operation of a contact round surgery of index 1, we remove two contact solid tori and reglue a contact thickened torus. Then the operation for the surgered manifold removing the glued contact thickened torus and regluing the removed contact solid tori recover the original contact manifold. The second operation is a contact round surgery of index 2 from the surgered manifold to the original one. In fact, from the definition in Subsection 4.1,

the solid tori are tubular neighborhoods of transverse knots, and the thickened torus is the invariant tubular neighborhood of a convex torus. The same property holds for contact round surgery of index 2. A contact round surgery of index 2 is operated after isotoping the given torus with Condition (4.2). The torus can be isotoped to a convex torus with no homotopically trivial dividing curve (see Subsection 4.2). Therefore, a contact round surgery of index 2 is also an operation concerning tubular neighborhoods of transverse knots and an invariant tubular neighborhood of a convex torus. Then it is reversible.

Then, in order to show that the given closed contact 3-manifold is obtained from a contact 3-sphere by contact round surgeries of index 1 and index 2, it is sufficient to show that a contact 3-sphere is obtained from the given closed contact 3-manifold by contact round surgeries of index 1 and index 2. Now, we construct a 3-dimensional sphere with a contact structure from the given closed contact 3-manifold. Let  $(M, \xi)$  be the given closed contact 3-manifold. From the argument above, there exists a sequence of round surgeries which makes  $M$  the 3-dimensional sphere. By the same argument as in Subsection 6.1, this sequence with some detours can be operated as direct contact round surgeries. Thus we obtain a 3-dimensional sphere with some contact structure.

**Step 2:** We discuss contact round surgeries between the standard contact structure and any other contact structure on the 3-dimensional sphere  $S^3$ . From Theorem 2.3, in each homotopy class as plane fields other than the class of the standard contact structure, there exists a unique overtwisted contact structure up to isotopy. In the class of the standard structure, there exists another overtwisted contact structure. Recall that a simple Lutz twist changes the homotopy class of the contact structure under consideration (see Subsection 5.1). Then, it is known that all overtwisted contact structures on  $S^3$  which are not homotopic to the standard one are constructed from the standard structure by the Lutz twists (see [L], [El3]). The overtwisted structure homotopic to the standard contact structure is obtained from the standard structure by a full Lutz twist which is a consecutive two simple Lutz twists along the same transverse knot. As we show in Subsection 5.2, the simple Lutz twist is realized by a sequence of contact round surgeries (Theorem 5.1). Therefore, all contact structures on the 3-dimensional sphere  $S^3$  are obtained from the standard contact structure by contact round surgeries.

Thus Theorem B has been proved.  $\square$

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